Finite-size scaling of directed percolation above the upper critical dimension

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We consider analytically as well as numerically the finite-size scaling behavior in the stationary state near the nonequilibrium phase transition of directed percolation within the mean field regime, i.e., above the upper critical dimension. Analogous to equilibrium, usual finite-size scaling is valid below the upper critical dimension, whereas it fails above. Performing a momentum analysis of associated path integrals we derive modified finite-size scaling forms of the order parameter and its higher moments. The results are confirmed by numerical simulations of corresponding high-dimensional lattice models.

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Critical phenomena such as second order phase transitions are characterized by singularities causing a discontinuous behavior of various quantities at the transition point (e.g., the specific heat, the susceptibility, and the correlation length). These singularities are described by power laws defining the well-known critical exponents. Studying the phase transition of a given system, the aim of the investigation is to identify the set of critical exponents which characterizes (together with certain universal scaling functions) the so-called universality class. Since most systems are not analytically tractable their critical behavior is often investigated via numerical methods, for example Monte Carlo simulations or transfer matrix calculations. In these cases, the obtained data are limited to finite system sizes. Therefore finite-size scaling (FSS) is widely used to extrapolate to the behavior of the infinite systems. In particular, FSS is an efficient method to determine the critical exponents and provides certain universal scaling functions, i.e., it allows to identify the universality class (see Refs. [1,2] for reviews).

According to the phenomenological FSS theory [3], a finite system size L results in a rounding and shifting of the singularities. Furthermore, it is assumed that finite-size effects are controlled sufficiently close to the critical point by the ratio L/ξ_∞. Here, ξ_∞ denotes the spatial correlation length of the infinite system. Approaching the transition point, the correlation length diverges as $\xi_\infty \propto |\tau-\tau_c|^{-\nu}$, with the critical exponent ν and where τ is a temperaturelike variable which describes the distance to the critical point. Finite-size effects are negligible for $L \gg \xi_\infty$ (i.e., $L|\tau-\tau_c|^\nu \gg 1$). Otherwise, they are relevant, i.e., rounding and shifting effects occur if $L < \xi_\infty$. It is known in equilibrium that the hypothesis of the fundamental role of the ratio L/ξ_∞ is valid only below the so-called upper critical dimension d_c (see Ref. [4] for a recent review). Above this marginal dimension, mean field theories provide exact values of the critical exponents as well as of the scaling functions. Renormalization group treatments show that the failure of usual FSS within the mean field regime is related to variables (scaling fields) which become dangerously irrelevant for $d > d_c$ [5]. Dangerous irrelevant variables affect the scaling behavior qualitatively and furthermore cause the breakdown of hyperscaling laws which connect the critical exponents to the spatial dimension d. Investigations of this breakdown of usual finite size scaling trace back to [6]. After controversial discussions (see e.g., [7] and referenced therein) the problem was recently resolved [8] and a convincing agreement between numerical data and field theoretical results in achieved [8,9].

Compared to the equilibrium situation less is known in the case of nonequilibrium phase transitions. Therefore we consider in this work the absorbing phase transition of directed percolation (DP) as an exemplification. According to its robustness and ubiquity (including critical phenomena in physics, biology, epidemiology, as well as catalytic chemical reactions) DP is recognized as the paradigm of nonequilibrium phase transitions into absorbing states (see Ref. [10] for a readable review). Although an analytical solution is still lacking, DP plays a comparable role in the realm of nonequilibrium phase transitions as the Ising model in equilibrium. Previous investigations of FSS of DP focus to the absorbing phase below d_c, where $d_c=4$ [11]. Here, we are interested in finite-size properties above d_c. In particular, we study the steady state scaling behavior of finite systems in the active phase which is maintained by a homogeneous source. Using a momentum space analysis of path integrals associated to the field theoretical formulation of DP, we derive FSS exponents and universal scaling functions. Analogous to equilibrium, we demonstrate that usual FSS has to be modified in order to describe the scaling behavior within the mean field regime. Additional numerical simulations confirm the field theoretical results. But in contrast to equilibrium we observe a convincing agreement between the lowest mode finite-size analysis and corresponding numerical results.

The asymptotic behavior of the DP universality class is described by a minimal stochastic Markovian process represented by the Langevin equation [12,13]

$$\lambda^{-1}\partial_t n = -\left(\tau + \frac{g}{2}n - \nabla^2\right)n + h + \zeta. \tag{1}$$

Here, the density of an active agent $n(\mathbf{r},t)$, defined on a mesoscopic (coarse grained) scale, corresponds to the order

parameter of the nonequilibrium phase transition. The control parameter of the transition τ attains its critical value in an infinite volume at τ_c. The homogeneous source h is conjugated to the order parameter and is usually implemented as a spontaneous creation of activity (see, e.g., Ref. [14]). For zero h, a finite positive density occurs above the transition point ($\tau<\tau_c$) whereas the absorbing vacuum state ($n=0$) is approached below the transition point. Furthermore, $\zeta(\mathbf{r},t)$ denotes the noise which accounts for fluctuations of the density $n(\mathbf{r},t)$. This zero-mean Gaussian noise represents fast degrees of freedom which were eliminated by a suitable coarse graining procedure. The noise correlator

$$\overline{\zeta(\mathbf{r},t)\zeta(\mathbf{r}',t')} = \lambda^{-1}g' n(\mathbf{r},t)\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') \tag{2}$$

is dictated by the existence of the absorbing state $n=0$.

Renormalization group techniques have been applied to determine the critical exponents and the universal scaling functions [11–13,15–17]. In that case path integral formulations are more adequate than the Langevin equation approach [18,19]. Stationary correlation functions as well as response functions can be determined by calculating path integrals with weight $\exp(-\mathcal{J})$, where the dynamic functional $\mathcal{J}$ describes the considered stochastic process. The following dynamic response functional [12,13]

$$\mathcal{J} = \int d^d r\, dt\, \lambda\left\{\tilde{n}\left(\lambda^{-1}\partial_t + (\tau - \nabla^2) + \frac{g}{2}(n-\tilde{n})\right)n - h\tilde{n}\right\} \tag{3}$$

is associated to the stochastic process defined by Eqs. (1) and (2). Here, $\tilde{n}(\mathbf{r},t)$ denotes the response field conjugated to the Langevin noise. Furthermore, the coupling constants g and g' are equated by an appropriate rescaling with the redundant parameter, K

$$\tilde{n}(\mathbf{r},t) \to K^{-1}\tilde{n}(\mathbf{r},t), \quad n(\mathbf{r},t) \to Kn(\mathbf{r},t), \quad h \to Kh. \tag{4}$$

The functional $\mathcal{J}$ is invariant under the time inversion (so-called rapidity reversal) $\tilde{n}(\mathbf{r},t) \leftrightarrow -n(\mathbf{r},-t)$ for vanishing (symmetry breaking) source h.

Using standard techniques known from equilibrium [6], it is possible to calculate size-dependent universal scaling functions as well as the involved critical exponents. We consider DP in a finite cubic geometry of linear size L with periodic boundary conditions and expand n and $\tilde{n}$ in complex exponential plane waves, e.g.,

$$n(\mathbf{r},t) = \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} n(\mathbf{q},t). \tag{5}$$

Each component of the wave vector takes only discrete values, precisely multiples of $2\pi/L$ including zero. Following Ref. [11], a dynamic free energy functional $\Sigma[\tilde{\Phi},\Phi]$ for the $\mathbf{q}=\mathbf{0}$ mode is constructed by decomposing the critical homogenous modes $\tilde{\Phi}(t)$, $\Phi(t)$, from their orthogonal noncritical complements $\tilde{\Psi}(\mathbf{r},t)$, $\Psi(\mathbf{r},t)$, e.g.,

$$n(\mathbf{r},t) = \Phi(t) + \Psi(\mathbf{r},t) \tag{6}$$

with $\Phi(t)=L^{-d}\int d^d r\, n(\mathbf{r},t)$. This leads to a decomposition of the response functional $\mathcal{J}=\mathcal{J}_0+\mathcal{J}_1$ with

$$\mathcal{J}_0 = \lambda L^d \int dt\left\{\tilde{\Phi}\left(\lambda^{-1}\partial_t + \tau + \frac{g}{2}(\Phi-\tilde{\Phi})\right)\Phi - h\tilde{\Phi}\right\}. \tag{7}$$

Now, $\tilde{\Psi}$ and Ψ are eliminated by a functional integration

$$e^{-\Sigma[\tilde{\Phi},\Phi]} = e^{-\mathcal{J}_0[\tilde{\Phi},\Phi]}\int \mathcal{D}[\tilde{\Psi},\Psi]e^{-\mathcal{J}_1[\tilde{\Psi},\Psi;\tilde{\Phi},\Phi]}. \tag{8}$$

The part $\mathcal{J}_1$ contributes to the leading scaling behavior for $d\leqslant d_c$ [11]. This will be revisited in a successional publication [20]. Here, we consider the mean field regime ($d>d_c$) where $\mathcal{J}_1$ provides, besides the shift of the control parameter from its mean field value ($\tau_c^{\mathrm{mf}}=0$) to $\tau_{c,L}=\tau_{c,\infty}+O(L^{2-d})<0$, corrections to the leading asymptotic scaling behavior. Hence we neglect $\mathcal{J}_1$ in the following but include the shift of the critical point to the infinite size value $\tau_c=\tau_{c,\infty}$ by the redefinition $\tau\to\tau-\tau_c$. Correlation functions of the order parameter Φ, $\langle\Pi_{\alpha=1}^k\Phi(t_\alpha)\rangle = G_k(\{t_\alpha\},\tau,h,L^d,\lambda,g)$, can be derived from path integrals with weight $\exp(-\Sigma)$. In that way, Eq. (7) and simple dimensional scaling leads to the parameter reduction in the correlation function

$$\begin{aligned} G_k(\{t_\alpha\},\tau,h,L^d,\lambda,g) &= g^{-k}F_k(\{\lambda t_\alpha\},\tau,gh,L^d/g^2) \\ &= L^{-kd/2}f_k(\{L^{-d/2}g\lambda t_\alpha\},L^{d/2}\tau/g,L^d h/g). \end{aligned} \tag{9}$$

Therefore it is convenient to define

$$\varphi(s) = L^{d/2}\Phi(t) \quad \text{and } s = gL^{-d/2}\lambda t. \tag{10}$$

Then Eq. (7) yields

$$\Sigma \approx \mathcal{J}_0 = \int ds\left\{\tilde{\varphi}\left(\partial_s + T + \frac{1}{2}(\varphi-\tilde{\varphi})\right)\varphi - H\tilde{\varphi}\right\}, \tag{11}$$

where the rescaled control parameter and the rescaled source

$$T = g^{-1}L^{d/2}\tau \quad \text{and } H = g^{-1}L^d h \tag{12}$$

are introduced. Note that the whole dynamic functional depends on the rescaled parameters T and H only. Furthermore, the rescaled parameters contain the irrelevant parameter g in a dangerous, i.e., singular, way.

The rescaled parameters [Eq. (12)] and Eq. (9) already contain a nontrivial result: As usual for critical phenomena, physical quantities of interest are described in terms of generalized homogenous functions. For example, the steady state order parameter $n=\langle\Phi\rangle$ and the steady state correlation length ξ obey for all $l>0$ the scaling forms (despite nonuniversal metric factors)

$$n = l^{\beta/\nu^*}\tilde{R}(\tau l^{-1/\nu^*}, h l^{-\Delta/\nu^*}, Ll), \tag{13}$$

$$\xi = l^{-1}\tilde{\Xi}(\tau l^{-1/\nu^*}, h l^{-\Delta/\nu^*}, Ll), \tag{14}$$

with the universal functions $\tilde{R}$ and $\tilde{\Xi}$ (analogous scaling functions are known from equilibrium [4,23]). Usual FSS forms involve the correlation length exponent ν whereas the above modified scaling forms contain the so far unknown exponent ν^*. For $d>d_c$, the order parameter exponent β and the field exponent Δ (often called gap exponent in equilib-

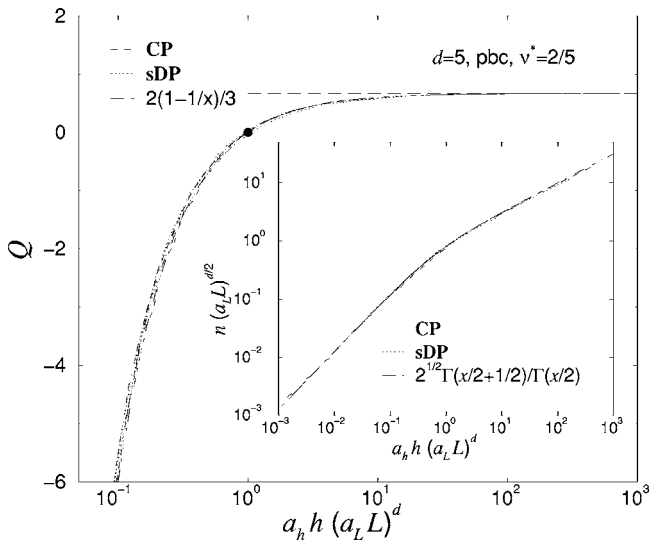

FIG. 1. The universal order parameter scaling function $\widetilde{R}_{\mathrm{pbc}}(0,x,1)$ (inset) and the universal fourth order ratio scaling function $\widetilde{Q}_{\mathrm{pbc}}(0,x,1)$ as a function of the rescaled field $a_h h(a_L L)^d$ at criticality for $d>d_c$. The analytically obtained scaling functions are in perfect agreement with numerical data of the five-dimensional contact process [CP, implemented on simple cubic lattices of size $L=4$, 8, 16, $\lambda_c=1.138\,46(11)$] and of the five-dimensional site-directed percolation process (sDP, implemented via the Domany-Kinzel automaton [24] on lattices of a generalized bcc-like structure [22] of linear size $L=8$, 16, 32, $p_c=0.0359\,725(2)$ [25]). Note that the numerical data already belong to the asymptotic scaling regime. In the case of the numerical data, nonuniversal metric factors a_h and a_L have been introduced in order to norm the universal scaling functions, i.e., $\widetilde{R}_{\mathrm{pbc}}(0,0,1)=1$ for the order parameter and $\widetilde{Q}_{\mathrm{pbc}}(0,1,1)=0$ (bold circle) for the ratio Q (see Refs. [21,22] for details).

rium) equal their mean field values $\beta=1$ and $\Delta=2$. Comparing Eqs. (9) and (12) to the scaling forms Eqs. (13) and (14) for $l=L^{-1}$ yields the FSS exponent for periodic boundary conditions

$$\nu^* = \frac{2}{d}. \tag{15}$$

Note that ν^* depends on the spatial dimension in contrast to the exponents β and Δ. More important, ν^* differs from the known mean field value of the correlation length exponent $\nu=1/2$ for $d>d_c=4$. Thus the FSS forms are not controlled by the ratio $L/\xi_\infty \propto L|\tau|^{\nu}$ but by $L|\tau|^{\nu^*}$. This scaling anomaly occurs within the mean field regime only and is regarded as breakdown of FSS. Furthermore, it can be interpreted as an appearance of an additional length scale, termed thermodynamic length scale l_∞, which diverges as $l_\infty \propto |\tau|^{-\nu^*}$ [4,23]. Similar to equilibrium, this length scale coincides with ξ_∞ below d_c, including $\nu=\nu^*$. Thus the exponent ν^* fulfills the hyperscaling relation $\nu^* d=2\beta+\gamma'$ in all dimensions.

Additionally to the critical exponent ν^* is it even possible to derive universal scaling functions, e.g., $\widetilde{R}(0,x,1)$. The dynamic functional Eq. (11) corresponds to the following Fokker-Planck equation [11]:

$$\partial_s P(\varphi,s) = \left\{ \partial_\varphi \left[\left(T + \frac{\varphi}{2} \right)\varphi - H \right] + \partial_\varphi^2 \frac{\varphi}{2} \right\} P(\varphi,s). \tag{16}$$

The stationary solution $P_0(\varphi)$

$$P_0(\varphi) = C\varphi^{2H-1} e^{-(2T+\varphi/2)\varphi} \tag{17}$$

can be normalized by an appropriate finite factor $C(H,T)$ for $H>0$. Straightforward calculations yield the moments at bulk criticality ($T=0$)

$$\langle \varphi^k \rangle = 2^{k/2}\Gamma(H+k/2)/\Gamma(H). \tag{18}$$

Thus the universal FSS functions of the order parameter $n=\langle\Phi\rangle=L^{-d/2}\langle\varphi\rangle$, of the order parameter fluctuations $\Delta n = L^d(\langle\Phi^2\rangle-\langle\Phi\rangle^2)=\langle\varphi^2\rangle-\langle\varphi\rangle^2$ as well as of the ratios $V=\langle\Phi^2\rangle/\langle\Phi\rangle^2-1$, $S=1-\langle\Phi^3\rangle/(3\langle\Phi\rangle\langle\Phi^2\rangle)$, $Q=1-\langle\Phi^4\rangle/(3\langle\Phi^2\rangle^2)$ are given by

$$n = \sqrt{\frac{2}{L^d}} \frac{\Gamma\left(\frac{x+1}{2}\right)}{\Gamma\left(\frac{x}{2}\right)} = L^{-d/2} \begin{cases} \sqrt{x}, & x \to \infty \\ \sqrt{\pi/2}x, & x \to 0, \end{cases} \tag{19}$$

$$\Delta n = x - 2\frac{\Gamma\left(\frac{x+1}{2}\right)^2}{\Gamma\left(\frac{x}{2}\right)^2} = \begin{cases} 1/2, & x \to \infty \\ x, & x \to 0, \end{cases} \tag{20}$$

$$V = \frac{x\Gamma\left(\frac{x}{2}\right)^2}{2\Gamma\left(\frac{x+1}{2}\right)^2} - 1 = \begin{cases} 1/2x, & x \to \infty \\ 2/\pi x, & x \to 0, \end{cases} \tag{21}$$

$$S = \frac{2}{3}\left(1 - \frac{1}{2x}\right), \quad Q = \frac{2}{3}\left(1 - \frac{1}{x}\right), \tag{22}$$

with the scaling argument $x=2H=2hL^d/g$. In contrast to equilibrium, the ratios V,S,Q are not finite at the critical point ($x\to 0$). This reflects the different nature of the zero order parameter phase in equilibrium and in absorbing phase transitions. A ratio that remains finite at criticality is obtained via

$$U(x) = \frac{2-3S(x)}{2-3Q(x)} = \frac{\langle\Phi^2\rangle\langle\Phi^3\rangle - \langle\Phi\rangle\langle\Phi^2\rangle^2}{\langle\Phi\rangle\langle\Phi^4\rangle - \langle\Phi\rangle\langle\Phi^2\rangle^2} = \frac{1}{2}. \tag{23}$$

We expect that this ratio is as useful for absorbing phase transition as the Binder cumulant Q is for equilibrium, i.e., its value at criticality characterizes the universality class. Preliminary numerical investigations below d_c yield $U_{d=1}=0.833$, $U_{d=2}=0.704$, and $U_{d=3}=0.61$ for $x\to 0$ [20].

The order parameter n and the ratio Q are shown in Fig. 1. Additionally to the above derived universal scaling functions we plot corresponding simulation results of the five-dimensional contact process (CP) as well as of the five-

dimensional site-directed percolation process (sDP). Both models belong to the DP universality class (see Ref. [10] and references therein). In contrast to conventional equilibrium simulation techniques, no steady state finite-size quantities are available for absorbing phase transitions at zero field. Close to the transition point, the systems will be soon trapped in the absorbing state without chance of escape. As recently pointed out in Ref. [21], the natural way to circumvent these difficulties is to perform simulations in nonzero field at criticality. Thus both the analytical results as well as the numerical simulations reflect that well-defined steady state quantities exist close to the critical point for $h>0$ only. As can be seen in Fig. 1, the data of the lattice models obey the modified FSS forms and the obtained scaling curves are in perfect agreement with the results of the continuum theory. A comment is worth making: In order to reach numerically the asymptotic scaling regime, the considered system sizes L have to exceed all intrinsic nonuniversal length scales L_0, i.e., $L \gg L_0$. The convincing agreement between the numerical and the field theoretical results indicates that L_0 is sufficiently small for the quantities Eqs. (19)–(22).

In contrast to, e.g., $Q(x)$, the ratio U exhibits a different behavior. To be precise, the leading order of U is no longer a function of the scaling argument $x=2hL^d/g$ within the mean field regime. Therefore nonuniversal corrections to scaling become dominant. Analytically, nonuniversal corrections to $U=1/2$ are obtained by incorporating the shift of the critical value $[\tau \to \tau - O(L^{(4-d)/2})]$. The results are confirmed numerically and will be published in a forthcoming paper [20]. Again, if the universal leading order of the order parameter moments $\langle \Phi^k \rangle$ is canceled for ratios such as U, a nonuniversal behavior occurs. Thus U is an appropriate quantity to investigate the relevance of corrections to scaling for $d>d_c$. In summary, convincing agreement is observed between the lowest mode finite-size analysis and corresponding numerical results. This is in contrast to the situation in equilibrium where it is known that the simplest lowest mode approach fails to describe the scaling behavior [8].

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[1] M. N. Barber, in *Finite-size Scaling in Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, New York, 1984), Vol. 8.
[2] J. L. Cardy, *Finite-size Scaling*, edited by J. L. Cardy (North-Holland, Amsterdam, 1988).
[3] M. E. Fisher and M. N. Barber, Phys. Rev. Lett. **28**, 1516 (1972).
[4] J. G. Brankov, D. M. Danchev, and N. S. Tonchev, in *Series in Modern Condensed Matter Physics* (World Scientific, Singapore, 2000), Vol. 9.
[5] M. E. Fisher, in *Renormalization Group in Critical Phenomena and Quantum Field Theory*, edited by D. J. Gunton and M. S. Green (Temple University, Philadelphia, 1974).
[6] E. Brézin, J. Phys. (France) **43**, 15 (1982); E. Brézin and J. Zinn-Justin, Nucl. Phys. B **257**, 867 (1985); J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3nd revised edition, (Clarendon, Oxford, 1996).
[7] E. Luijten and H. W. J. Blöte, Phys. Rev. Lett. **76**, 1557 (1996); H. W. J. Blöte and E. Luijten, Europhys. Lett. **38**, 565 (1997); X. S. Chen and V. Dohm, Eur. Phys. J. B **5**, 529 (1998); E. Luijten, K. Binder, and H. W. J. Blöte, Eur. Phys. J. B **9**, 289 (1999).
[8] X. S. Chen and V. Dohm,Phys. Rev. E **63**, 016113 (2000)
[9] D. Stauffer, Braz. J. Phys. **30**, 787 (2000).
[10] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[11] H. K. Janssen, B. Schaub, and B. Schmittmann, Z. Phys. B: Condens. Matter **71**, 377 (1988).
[12] H. K. Janssen, Z. Phys. B: Condens. Matter **42**, 151 (1981).
[13] H. K. Janssen, J. Stat. Phys. **103**, 801 (2001).
[14] S. Lübeck and R. D. Willmann, J. Phys. A **35**, 10205 (2002).
[15] S. P. Obukhov, Physica A **101**, 145 (1980).
[16] J. L. Cardy and R. L. Sugar, J. Phys. A **13**, L423 (1980).
[17] H. K. Janssen, Ü. Kutbay, and K. Oerding, J. Phys. A **32**, 1809 (1999).
[18] H. K. Janssen, Z. Phys. B **23**, 377 (1976); R. Bausch, H. K. Janssen, and H. Wagner, *ibid.* **24**, 113 (1976); H. K. Janssen, in *Dynamical Critical Phenomena and Related Topics*, Lecture Notes in Physics Vol. 104, edited by C. P. Enz (Springer, Heidelberg, 1979); H. K. Janssen, in *From Phase Transitions to Chaos*, edited by G. Györgyi, I. Kondor, L. Sasvári, and T. Tél (World Scientific, Singapore, 1992).
[19] C. De Dominicis, J. Phys. (Paris), Colloq. **37**, 247 (1976); C. De Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).
[20] H. K. Janssen and S. Lübeck (to be published).
[21] S. Lübeck and P. C. Heger, Phys. Rev. E **68**, 056102 (2003).
[22] S. Lübeck and R. D. Willmann, J. Stat. Phys. **115**, 1231 (2005).
[23] K. Binder, M. Nauenberg, V. Privman, and A. P. Young, Phys. Rev. B **31**, 1498 (1985).
[24] E. Domany and W. Kinzel, Phys. Rev. Lett. **53**, 311 (1984).
[25] P. Grassberger (private communication).